Hydrodynamics of weakly asymmetric exclusion with slow boundary

Pedro Capitão and Patrícia Gonçalves

Abstract In this article we discuss the hydrodynamic limit of the weakly asymmetric simple exclusion process, whose asymmetry is regulated by a factor N^γ with $\gamma \geq 1$, and in contact with stochastic reservoirs, which are regulated by two factors N^θ and N^δ for, respectively, the symmetric and asymmetric parts of the jump rate at the boundary. Depending on the strength of the asymmetry, that is on the parameter γ, we derive the heat equation (when $\gamma > 1$) as in the purely symmetric case studied in [1], or the viscous Burgers equation (when $\gamma = 1$). In both cases, the PDEs have several boundary conditions which depend on the range of the parameters δ and θ.

Key words: Exclusion process, weakly asymmetric rates, slow boundary, hydrodynamic limits

1 Introduction

Interacting particle systems are a class of Markov processes, which were introduced around 1970 (see [17]) and since then they have been widely studied, see [15, 16]. These systems are used as toy models in contexts such as statistical physics, neural networks, spread of infections and evolution of biological populations. Other than the exclusion process, which is our process of interest, examples of interacting particle systems include the stochastic Ising model, the voter model, the contact process and the zero-range process (see, for example, [15]).

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The main defining features of interacting particle systems are the following: they usually have state spaces of the form $\Omega = \{0,1\}^{\Lambda}$ or $\Omega = \mathbb{N}^{\Lambda}$, where Λ is a countable set, and they can be seen as a superposition of processes, each describing the movement of a single particle. These individual particle processes, with values in Λ, would behave like independent random walks, if it were not for the interaction mechanics. In this interpretation, a state of the process $\eta \in \Omega$ is seen as a configuration of particles and $\eta(x)$ represents the number of particles at site $x \in \Lambda$. Each model is characterized by its state space and the set of laws that dictate how particles interact. These laws determine the evolution of the system as a whole, which is always assumed to be Markovian.

The process we are going to study is a simple exclusion process on the discrete space $\Lambda_N = \{1,\dots,N-1\}$, for some $N \in \mathbb{N}$. This means that at most one particle can occupy each site (hence the state space is $\Omega_N = \{0,1\}^{\Lambda_N}$), each transition affects two sites and corresponds to the jump of a particle from one to the other, and particles can only jump to their nearest neighbours. This process can be thought of as a superposition of random walks, with the law for interaction being that jumps to sites that are already occupied are suppressed. For each possible jump, the time until that jump occurs is exponentially distributed. Furthermore, all these times are independent of each other and depend only on the current configuration. This ensures that the system is a Markov process. Note that these exponentially distributed random variables do not necessarily have all the same rate.

The particular model we want to study is the weakly asymmetric simple exclusion process (abbreviated WASEP) in contact with reservoirs. The particles move on $\Lambda_N = \{1,\dots,N-1\}$, which is called the bulk, and the state space of the process is $\Omega_N = \{0,1\}^{\Lambda_N}$. The sites 0 and N act as reservoirs: they have an infinite number of particles and can add or remove particles from the sites in the bulk immediately next to them. The process is weakly asymmetric in the sense that the rate at which particles jump to the left or right is not the same (jumps to the right are slightly preferred and happen at rate $1+\frac{E}{N^\gamma}$, while to the left they happen at rate 1), but this difference between the two rates becomes smaller as N increases. We restrict the parameter γ to be greater or equal to 1. A representation of these dynamics is given in Figure 1.

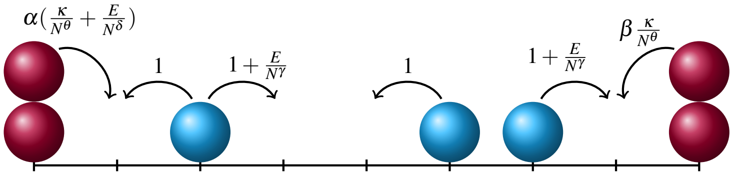

Fig. 1 Possible transitions in the WASEP with reservoirs.

Our goal is to study the density of particles in the limit as the scaling parameter N tends to infinity. More precisely, we want to show that the empirical measure, which is a random measure on $[0,1]$ weighted by the number of particles, converges to an absolutely continuous deterministic measure whose density is the solution of some partial differential equation, called the *hydrodynamic equation*. This is the equation that governs the macroscopic evolution of the density of the system. According to the parameters of the model, we will identify the equations that should be obtained for each one of their values. Depending on the value of γ the nature of the PDE changes, more precisely, for $\gamma > 1$ we derive the heat equation while for $\gamma = 1$ we derive the viscous Burgers equation. Depending on the values of the parameters that regulate the intensity of the reservoirs we obtain these equations with several boundary conditions. We give the description of the hydrodynamics for the cases $\theta \geq 0$ and $\delta \geq 1$ and the case $\delta = \theta \in [0,1)$. The remaining cases are left for a future work.

We employ the entropy method developed in [13] and the main reason for the method not to work in the case we left open is the fact that we do not have enough information on the stationary measure of the model to derive good bounds to compare the Dirichlet form operator and the carré du champ operator, which is crucial when proving replacement lemmas. We remark that the case of symmetric jumps was analyzed in [1] where the heat equation was derived with several boundary conditions and corresponds to our case when $E = 0$ (no asymmetry). The case when jumps are arbitrarily large and given by a symmetric transition probability $p(\cdot)$ with finite variance was studied in [2] and the case of infinite variance was studied both in [3, 4].

The motivation for the study of the model we consider in this article is to have intuition for the case where long jumps are present, as in the aforementioned articles, but considering a transition probability which is weakly asymmetric. The goal will be to derive the fractional viscous Burgers equation with several (fractional) boundary conditions. This is left for a future work.

Once the hydrodynamic limit is established, there are several possiblities to attack as, for example, the Fick's Law, the large deviation principle, the fluctuations, among others. Let us just briefly discuss one of the paths we want to pursue: the fluctuations. We observe that for our model in the case $\kappa = 1$, $\gamma = 1$, $\theta = 0$, $\delta = 1$ the equilibrium fluctuations were established in [6], while the non-equilibrium fluctuations were analysed in [11]. When the asymmetry increases, the equilibrium fluctuations in the case $\kappa = 1$, $\gamma \in [\frac{1}{2}, +\infty)$, $\theta = 0$, $\delta = \gamma$ and $\alpha = \beta$, were derived in [12], and for $\gamma > \frac{1}{2}$ the limit is an Ornstein-Uhlenbeck process; while for $\gamma = 1/2$ the limit is an energy solution of the stochastic Burgers equation, both with Dirichlet boundary conditions. As a continuation of our study we intend to analyse the remaining cases in a future work, specially for the case of a strong asymmetric regime.

This article is organized as follows. In Section 2 we present the models that we study, the hydrodynamic equations that we derive and the respective notions of weak solutions, and we state our main result, the hydrodynamic limit. We assume that the PDEs we obtain have a unique weak solution, according to Definitions 2 and 3, since the equations are quite classical in the PDE's literature. In Section 3 we give a heuristic argument on how to derive the notions of weak solutions for each one of the regimes that we study. This is done through auxiliary martingales that can be associated to Markov Processes by Dynkin's formula, which gives a discretization of the notion of weak solution. Then one just has to justify that the limit in N converges to the integral solution given either in (3) or (5). In Section 4 we prove tightness of the sequence of empirical measures. In Section 5 we prove all the replacement lemmas that we need along the arguments, in order to recognize the limit of the martingales as the solution of the respective PDE.

2 Statement of results

2.1 The models

Fix the following parameters: $E,\kappa>0$, $\alpha,\beta\in(0,1)$, $\gamma\geq 1$, and $\theta,\delta\geq 0$. Let N be a natural number. Denote by $\{\eta_t : t\geq 0\}$, the one-dimensional, boundary driven, weakly asymmetric simple exclusion process with state space $\Omega_N=\{0,1\}^{\Lambda_N}$, where $\Lambda_N=\{1,\dots,N-1\}$. The configurations of the state space are denoted by the symbol η, so that $\eta(x)=1$ if site $x\in\Lambda_N$ is occupied for the configuration η and $\eta(x)=0$ if site x is empty. The infinitesimal generator of the Markov process $\{\eta_t : t\geq 0\}$ is denoted by $\mathscr{L}_N$ and acts on functions $f:\Omega_N\to\mathbb{R}$ as $\mathscr{L}_N=\mathscr{L}_N^B+\mathscr{L}_N^L+\mathscr{L}_N^R$, where the terms on the right are the generators corresponding to the dynamics of the bulk, the left boundary, and the right boundary, respectively:

$$\begin{aligned}
(\mathscr{L}_N^B f)(\eta) &= \sum_{x=1}^{N-2} c_{x,x+1}(\eta)\,\{f(\eta^{x,x+1})-f(\eta)\}\,,\\
(\mathscr{L}_N^L f)(\eta) &= c_{0,1}(\eta)\,\{f(\eta^{1})-f(\eta)\}\,,\\
(\mathscr{L}_N^R f)(\eta) &= c_{N-1,N}(\eta)\,\{f(\eta^{N-1})-f(\eta)\}\,.
\end{aligned}\tag{1}$$

where, for $1\leq x\leq N-2$,

$$\begin{aligned}
c_{x,x+1}(\eta) &= \Big(1+\frac{E}{N^\gamma}\Big)\eta(x)\,[1-\eta(x+1)] \;+\; \eta(x+1)\,[1-\eta(x)]\,,\\
c_{0,1}(\eta) &= \Big(\frac{\kappa}{N^\theta}+\frac{E}{N^\delta}\Big)\,\eta(0)\,[1-\eta(1)]+\frac{\kappa}{N^\theta}\eta(1)\,[1-\eta(0)]\,,\\
c_{N-1,N}(\eta) &= \Big(\frac{\kappa}{N^\theta}+\frac{E}{N^\delta}\Big)\eta(N-1)\,[1-\eta(N)] \;+\; \frac{\kappa}{N^\theta}\eta(N)\,[1-\eta(N-1)],
\end{aligned}$$

with the convention that $\eta(0)=\alpha$ and $\eta(N)=\beta$. In these formulas, for $1\leq x\leq N-2$, $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x+1)$:

$$(\eta^{x,x+1})(y)=\begin{cases}\eta(x+1), & y=x\,,\\ \eta(x), & y=x+1\,,\\ \eta(y), & y\neq x,x+1\,,\end{cases}$$

while for $x \in \{1, N-1\}$, η^x is the configuration obtained by flipping the occupation variable $\eta(x)$, that is $(\eta^x)(y) = \eta(y)\mathbf{1}_{y \neq x} + (1-\eta(y))\mathbf{1}_{y=x}$.

From now on we fix a finite time horizon $[0,T]$. The trajectories of our process $\{\eta_t : t \geq 0\}$ are elements of the Skorohod space $\mathscr{D}([0,T], \Omega_N)$, which is defined as the set of all functions $\eta_\cdot : [0,T] \to \Omega_N$ that are right continuous with left limits.

We observe that, since our process is a finite state Markov process, there exists a unique stationary measure. For $\alpha = \beta = \rho$ and $E = 0$ or $\gamma = \delta$ the invariant measure is the Bernoulli product measure ν^N_ρ with marginals

$$\nu^N_\rho\{\eta : \eta(x) = 1\} = \rho,$$

for $x \in \Lambda_N$. In fact, one can show that, when $\alpha = \beta = \rho$ and $E = 0$ this measure is also reversible. Nevertheless, for other values of the parameters, we do not have information about the stationary measure, except some partial description that can be given by the matrix ansatz method, but which, in fact, we do not use in this article.

2.2 *Hydrodynamic equations*

We denote by $\langle \cdot, \cdot \rangle_\mu$ the inner product in $L^2([0,1])$ with respect to a measure μ defined in $[0,1]$ and $\| \cdot \|_{L^2(\mu)}$ is the corresponding norm. When μ is the Lebesgue measure we write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_{L^2}$ for the corresponding norm.

We denote by $C^{m,n}([0,T] \times [0,1])$ the set of functions defined on $[0,T] \times [0,1]$ that are m times continuously differentiable on the first variable and n times continuously differentiable on the second variable. For a function $G := G_t(q) \in C^{m,n}([0,T] \times [0,1])$ we denote by $\partial_t G$ its derivative with respect to the time variable t and by ∇G and ΔG its first and second derivatives, respectively, with respect to the space variable q. The supremum norm is denoted by $\| \cdot \|_\infty$. Finally, $C^{m,n}_0([0,T] \times [0,1])$ is the set of functions $G \in C^{m,n}([0,T] \times [0,1])$ such that for any time t the function G_t vanishes at the boundary, that is, $G_t(0) = G_t(1) = 0$.

Now we want to define the space on which the solutions of the hydrodynamic equations will live, namely the Sobolev space $\mathscr{H}_1$ on $[0,1]$. For that purpose, we define the semi inner-product $\langle \cdot, \cdot \rangle_1$ on the set $C^\infty([0,1])$ by

$$\langle G, H \rangle_1 = \int_0^1 (\nabla G)(q)\,(\nabla H)(q)\,dq,$$

for $G, H \in C^\infty([0,1])$ and the corresponding semi-norm is denoted by $\| \cdot \|_1$.

Definition 1. The Sobolev space $\mathscr{H}_1$ on $[0,1]$ is the Hilbert space defined as the completion of $C^\infty([0,1])$ for the norm $\| \cdot \|^2_{\mathscr{H}_1} := \| \cdot \|^2_{L^2} + \| \cdot \|^2_1$. Its elements coincide a.e. with continuous functions. The space $L^2([0,T], \mathscr{H}_1)$ is the set of measurable functions $f : [0,T] \to \mathscr{H}_1$ such that $\int_0^T \|f_s\|^2_{\mathscr{H}_1} ds < \infty$.

We can now give the definition of the weak solutions of the hydrodynamic equations that will be derived. In what follows, $\rho_0 : [0,1] \to [0,1]$ is a measurable function and it is the initial condition of all the partial differential equations that we define below.

Definition 2. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the viscous Burgers equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t \rho(t,q) = \Delta \rho(t,q) - \mathfrak{e}\, \nabla \sigma(\rho(t,q))\,, \quad t \in [0,T]\,, \quad q \in [0,1]\,, \\ \rho(t,0) = a\,, \quad \rho(t,1) = b\,, \quad t \in [0,T], \\ \rho(0,q) = \rho_0(q)\,, \quad q \in [0,1]\,, \end{cases} \tag{2}$$

if the following two conditions hold:

- $\rho \in L^2([0,T], \mathscr{H}_1)$;
- ρ satisfies the weak formulation

$$\begin{aligned} &\int_0^1 \rho_t(q) G_t(q)\, dq - \int_0^1 \rho_0(q) G_0(q)\, dq - \int_0^t \int_0^1 \rho_s(q) \Big(\Delta + \partial_s \Big) G_s(q)\, dq\, ds \\ &- \int_0^t \int_0^1 \mathfrak{e} \sigma(\rho_s(q)) \nabla G_s(q) dq ds + \int_0^t \big(b \nabla G_s(1) - a \nabla G_s(0) \big)\, ds = 0, \end{aligned} \tag{3}$$

for all $t \in [0,T]$ and any function $G \in C_0^{1,2}([0,T] \times [0,1])$.

Above $\sigma(\rho)$ represents the mobility and it is given by $\sigma(\rho) = \rho(1-\rho)$.

Observe that when $\mathfrak{e} = 0$ the equation above reduces to the heat equation.

Definition 3. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the viscous Burgers equation with Robin boundary conditions

$$\begin{cases} \partial_t \rho(t,q) = \Delta \rho(t,q) - \mathfrak{e}\, \nabla \sigma(\rho(t,q))\,, \quad t \in [0,T]\,, \quad q \in [0,1]\,, \\ \nabla \rho(t,0) = a\rho(t,0) + b + \mathfrak{e}\sigma(\rho(t,0))\,, \quad t \in [0,T], \\ \nabla \rho(t,1) = c\rho(t,1) + d + \mathfrak{e}\sigma(\rho(t,1))\,, \quad t \in [0,T], \\ \rho(0,q) = \rho_0(q)\,, \quad q \in [0,1]\,, \end{cases} \tag{4}$$

if the following two conditions hold:

- $\rho \in L^2([0,T], \mathscr{H}_1)$;
- ρ satisfies the weak formulation

$$\begin{aligned} &\int_0^1 \rho_t(q) G_t(q)\, dq - \int_0^1 \rho_0(q) G_0(q)\, dq - \int_0^t \int_0^1 \rho_s(q) \Big(\Delta + \partial_s \Big) G_s(q)\, dq\, ds \\ &- \int_0^t \int_0^1 \mathfrak{e} \sigma(\rho_s(q)) \nabla G_s(q)\, dq\, ds + \int_0^t \rho_s(1) \nabla G_s(1)\, ds - \int_0^t \rho_s(0) \nabla G_s(0)\, ds \\ &- \int_0^t \big(c\rho_s(1) + d \big) G_s(1)\, ds + \int_0^t \big(a\rho_s(0) + b \big) G_s(0)\, ds = 0, \end{aligned} \tag{5}$$

for all $t \in [0,T]$ and any function $G \in C^{1,2}([0,T] \times [0,1])$.

The proof of the uniqueness of weak solutions in the case $\mathfrak{e} = 0$, which corresponds to the heat equation with Dirichlet, linear Robin or Neumann boundary conditions can be seen in Section 7 of [1]. For completeness we included in the Appendix the proof of uniqueness of weak solutions in the case $\mathfrak{e} \neq 0$ for Dirichlet or Neumann boundary conditions. Nevertheless, we assume the uniqueness of weak solutions for Robin boundary conditions in the case $\mathfrak{e} \neq 0$, but we believe that the proof could be adapted from the one in Section 7 of [5]. We leave this for a future work.

2.3 Hydrodynamic Limit

In this section we want to state the hydrodynamic limit of the process $\{\eta_{tN^2} : t \geq 0\}$ with state space Ω_N and infinitesimal generator $N^2\mathscr{L}_N$, where $\mathscr{L}_N$ is as defined in (1). Note that we are accelerating time by a factor of N^2. Let $\mathscr{M}^+$ be the space of positive measures on $[0,1]$ with total mass bounded by 1 equipped with the weak topology. For any configuration $\eta \in \Omega_N$ we define the empirical measure $\pi^N(\eta,dq)$ on $[0,1]$ by

$$\pi^N(\eta,dq) = \frac{1}{N-1}\sum_{x\in\Lambda_N} \eta(x)\delta_{\frac{x}{N}}(dq),$$

where δ_a is a Dirac mass on $a \in [0,1]$, and

$$\pi^N_t(\eta_\cdot,dq) := \pi^N(\eta_{tN^2},dq).$$

This measure gives weight $\frac{1}{N}$ to each occupied site of the configuration η.

We denote by $\mathbb{P}_{\mu_N}$ the probability measure on the Skorohod space $\mathscr{D}([0,T],\Omega_N)$ induced by the Markov process $\{\eta_{tN^2} : t \geq 0\}$ and initial distribution μ_N on Ω_N, and we denote by $\mathbb{E}_{\mu_N}$ the expectation with respect to $\mathbb{P}_{\mu_N}$. Now let $\{\mathbb{Q}_N\}_{N\geq 1}$ be the sequence of probability measures on $\mathscr{D}([0,T],\mathscr{M}^+)$ induced by the Markov process $\{\pi^N_t : t \geq 0\}$ and by $\mathbb{P}_{\mu_N}$.

At this point we need to fix an initial profile $\rho_0 : [0,1] \to [0,1]$ which is measurable and an initial distribution $\mu_N \in \Omega_N$. We are going to consider the following set of initial measures:

Definition 4. A sequence of probability measures $\{\mu_N\}_{N\geq 1}$ in Ω_N is associated with the profile $\rho_0(\cdot)$ if for any continuous function $G : [0,1] \to \mathbb{R}$ and any $\varepsilon > 0$

$$\lim_{N\to\infty} \mu_N\Big(\eta \in \Omega_N : \Big|\frac{1}{N-1}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\eta(x) - \int_0^1 G(q)\rho_0(q)dq\Big| > \varepsilon\Big) = 0.$$

Our main result is summarized in the following theorem.

Theorem 1. *Let $\rho_0 : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated with $\rho_0(\cdot)$. Then, for any $t \in [0,T]$ and any $\varepsilon > 0$,*

$$\lim_{N\to\infty}\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|\frac{1}{N-1}\sum_{x=1}^{N-1}G\left(\tfrac{x}{N}\right)\eta_{tN^2}(x)-\int_0^1 G(q)\rho_t(q)dq\Big|>\varepsilon\Big)=0,$$

where $\rho_t(\cdot)$ *is:*

- *for* $\gamma=1$ *the unique weak solution of*
  1. *for* $\theta>1$ *and* $\delta>1$, *the PDE* (4) *with* $\mathfrak{e}=E$ *and* $a=b=c=d=0$.
  2. *for* $\theta=1$ *and* $\delta>1$, *the PDE* (4) *with* $\mathfrak{e}=E$, $a=\kappa$, $b=-\kappa\alpha$, $c=-\kappa$ *and* $d=\kappa\beta$.
  3. *for* $\delta=1$ *and* $\theta>1$, *the PDE* (4) *with* $\mathfrak{e}=E$, $a=E\alpha$, $b=-E\alpha$, $c=-E(1-\beta)$ *and* $d=0$.
  4. *for* $\theta=\delta=1$, *the PDE* (4) *with* $\mathfrak{e}=E$, $a=\kappa+E\alpha$, $b=-(\kappa+E)\alpha$, $c=-(\kappa+E(1-\beta))$ *and* $d=\kappa\beta$.
  5. *for* $\theta=\delta\in[0,1)$, *the PDE* (2) *with* $\mathfrak{e}=E$, $a=\frac{\alpha(\kappa+E)}{\kappa+\alpha E}$ *and* $b=\frac{\kappa\beta}{\kappa+E(1-\beta)}$.
  6. *for* $\theta\in[0,1)$ *and* $\delta\geq 1$, *the PDE* (2) *with* $\mathfrak{e}=E$, $a=\alpha$ *and* $b=\beta$.
- *for* $\gamma>1$ *the unique weak solution of heat equation with the same boundary conditions as in the case* $\gamma=1$.

Above, the function G belongs to the space of test functions appearing in the definitions of the weak solutions of the corresponding PDEs. The partial differential equations corresponding to the values of the parameters θ *and* δ *are summarized in Figure 2.*

We note that above we obtained in the first four cases, for $\gamma=1$ (resp. $\gamma>1$) the viscous Burgers equation (resp. heat equation) with non-linear Robin boundary conditions (resp. linear Robin or Neumann boundary conditions); and in the last two cases, the boundary conditions are of Dirichlet type.

Remark 1. We observe that, by changing the value of E to $\tilde{E}$ only on the rates $c_{0,1}(\eta)$ and changing E to E' only on the rates $c_{N-1,N(\eta)}$ we would obtain the same PDEs as above, except that on the boundary conditions, we would replace in both a and b the value E by $\tilde{E}$ and in both c and d we would replace the value E by E'.

The analysis of the region $\delta\in[0,1)$ and $\theta\geq 0$ (apart the red line in the previous figure) is left for a future work. Nevertheless, we conjecture that the green zone should continue up to the red line (the region in light green) and the remaining region (the region in violet) should correspond to the PDE (2) with $a=1$ and $b=0$ and with $\mathfrak{e}=0$ when $\gamma>1$ and $\mathfrak{e}=E$ when $\gamma=1$.

The proof of last theorem follows the usual approach of convergence in distribution of stochastic processes: we have to prove tightness of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ (which is done in Section 4) from where we conclude that the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ has a subsequence that converges weakly to some measure that we denote by $\mathbb{Q}$. Then we characterize this measure by showing that it is supported on trajectories of measures that satisfy $\pi_t(dq)=\rho_t(q)dq$ where $\rho_t(q)$ is the unique weak solution

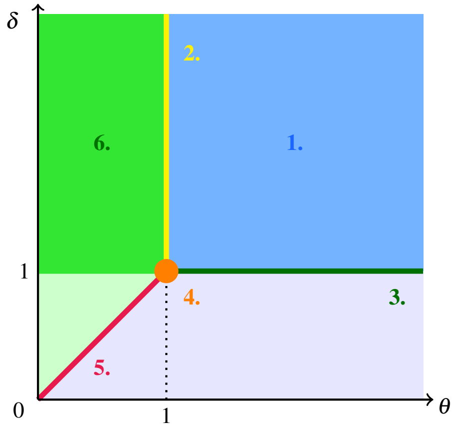

Fig. 2 Boundary conditions for the viscous Burgers equation (when $\gamma = 1$) and for the heat equation (when $\gamma > 1$) as functions of the model parameters θ and δ. The numbering refers to the equations above. The study of the regions in light green and violet is left for future work.

of the hydrodynamic equation. These two results combined give the convergence of $\{\mathbb{Q}_N\}_{N\geq 1}$ to $\mathbb{Q}$, as $N \to \infty$. Showing that the limit point $\mathbb{Q}$ is concentrated on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure is a consequence of the fact that our process is an exclusion process (we refer the reader to [14] for a proof), and the proof that the density $\rho_t(\cdot)$ is a weak solution of the hydrodynamic equation goes through the help of auxiliary martingales associated to the empirical measure, this is done in Section 3. We only present the heuristic argument of this derivation, but since we prove all the replacement lemmas (see Section 5) that are necessary, the rigorous proof is simple and is left to the reader. From the uniqueness of the weak solutions of the equations, we conclude that $\{\mathbb{Q}_N\}_{N\geq 1}$ has a unique limit point $\mathbb{Q}$.

3 Heuristics for hydrodynamic equations

In this section we give the main ideas which are behind the identification of limit points as weak solutions of the partial differential equations given in Section 2.2. Now we argue that the density $\rho_t(\cdot)$ is a weak solution of the corresponding hydrodynamic equation for each regime of the parameters γ, θ, δ. In order to prove that $\rho_t(\cdot)$ satisfies the weak formulation we use auxiliary martingales associated to the Markov process $\{\eta_{tN^2} : t \geq 0\}$. For that purpose, fix a function $G \in C^{1,2}([0,T] \times [0,1])$. By Dynkin's formula, see for example Lemma 5.1 of [14],

$$M_t^N(G) = \langle \pi_t^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^t (N^2 \mathscr{L}_N + \partial_s)\langle \pi_s^N, G\rangle \, ds \tag{6}$$

is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$, where for each $t \geq 0$, $\mathscr{F}_t := \sigma(\eta_s : s < t)$. By expanding the last term above and using the notation $\nabla_N^+ G_s(\frac{x}{N}) := N(G_s(\frac{x+1}{N}) - G_s(\frac{x}{N}))$, $\nabla_N^- G_s(\frac{x}{N}) := N(G_s(\frac{x}{N}) - G_s(\frac{x-1}{N}))$, $\Delta_N G_s(\frac{x}{N}) := N^2(G_s(\frac{x+1}{N}) - 2G_s(\frac{x}{N}) + G_s(\frac{x-1}{N}))$, we get the following expression:

$$\begin{aligned}
M_t^N(G) = {} & \langle \pi_t^N, G_t\rangle - \langle \pi_0^N, G_0\rangle - \int_0^t \langle \pi_s^N, \Delta_N G_s\rangle\, ds - \int_0^t \langle \pi_s^N, \partial_s G_s\rangle\, ds \\
& - E\int_0^t \frac{N^{1-\gamma}}{N-1}\sum_{x=1}^{N-2} \nabla_N^+ G_s(\tfrac{x}{N})\eta_{sN^2}(x)(1-\eta_{sN^2}(x+1))\, ds \\
& - \int_0^t \frac{N}{N-1}\nabla_N^+ G_s(\tfrac{1}{N})\eta_{sN^2}(1)\, ds + \int_0^t \frac{N}{N-1}\nabla_N^- G_s(\tfrac{N-1}{N})\eta_{sN^2}(N-1)\, ds \\
& - \frac{N^2}{N-1}\int_0^t G_s(\tfrac{1}{N})\Big\{\frac{\kappa}{N^\theta}(\alpha - \eta_{sN^2}(1)) + \frac{E}{N^\delta}\alpha(1-\eta_{sN^2}(1))\Big\}\, ds \\
& - \frac{N^2}{N-1}\int_0^t G_s(\tfrac{N-1}{N})\Big\{\frac{\kappa}{N^\theta}(\beta - \eta_{sN^2}(N-1)) - \frac{E}{N^\delta}(1-\beta)\eta_{sN^2}(N-1)\Big\}\, ds.
\end{aligned} \tag{7}$$

From this expression we will deduce the notion of weak solution of the equations presented above for several regimes of the parameters. We do not present the complete proof but we only highlight what are the ingredients one needs to obtain the results. First we observe that from Section 4 we see that $\mathbb{E}_{\mu_N}[(M_t^N(G))^2]$ vanishes as $N \to \infty$. Now we analyse the remaining terms. For that purpose we introduce some notation. We define $\Lambda_N^\varepsilon = \{1+\varepsilon N, \cdots, N-1-\varepsilon N\}$ and denote by $\overrightarrow{\eta}_{sN^2}^{\varepsilon N}(x)$ (resp. $\overleftarrow{\eta}_{sN^2}^{\varepsilon N}(x)$) the empirical density in the box of size εN, which is given on $x \in \Lambda_N^\varepsilon$ by

$$\overrightarrow{\eta}_{sN^2}^{\varepsilon N}(x) = \frac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\eta_{sN^2}(y) \qquad \Big(\text{resp. } \overleftarrow{\eta}_{sN^2}^{\varepsilon N}(x) = \frac{1}{\varepsilon N}\sum_{y=x-\varepsilon N+1}^{x}\eta_{sN^2}(y)\Big). \tag{8}$$

Note that above, εN should be understood as $\lfloor \varepsilon N\rfloor$ and that $\overrightarrow{\eta}_{sN^2}^{\varepsilon N}(x) = \langle \pi_{sN^2}^N, \iota_\varepsilon^x\rangle$, where $\iota_\varepsilon^x(u) = \frac{1}{\varepsilon}\mathbf{1}_{(\frac{x}{N},\frac{x}{N}+\varepsilon)}(u)$ and analogously for the definition of the left average. Heuristically, $\langle \pi_{sN^2}^N, \iota_\varepsilon^x\rangle$ converges, when $N\to\infty$, to $\langle \pi_s, \iota_\varepsilon^x\rangle = \int_0^1 \rho_s(u)\iota_\varepsilon^x(u)\,du$, where $\rho_s(\cdot)$ is the density profile that we want to characterize. Then, by taking the limit as $\varepsilon \to 0$ we obtain that $\langle \pi_s, \iota_\varepsilon^x\rangle$ converges to $\rho_s(\frac{x}{N})$. From the observation above we say that $\overrightarrow{\eta}_{sN^2}^{\varepsilon N}(x) \sim \rho_s(\frac{x}{N})$.

Now observe that a simple computation shows that the fifth term on the RHS of (7) vanishes if $\gamma > 1$, while for $\gamma = 1$, a simple application of Lemma 6 twice shows that we can replace that term by

$$-E\int_0^t \frac{1}{N-1}\sum_{x=1}^{N-2}\nabla_N^+ G_s(\tfrac{x}{N})\overleftarrow{\eta}_{sN^2}^{\varepsilon N}(x)(1-\overrightarrow{\eta}_{sN^2}^{\varepsilon N}(x+1))\, ds.$$

Indeed, to do that observe that the fifth term on the RHS of (7) can be written as

$$-E\int_0^t \frac{1}{N-1}\sum_{x\in\Lambda_N^\varepsilon}\nabla_N^+ G_s(\tfrac{x}{N})\,\eta_{sN^2}(x)(1-\eta_{sN^2}(x+1))\,ds+O(\varepsilon)$$

and now one applies Lemma 6 once to replace $\eta(x)$ by the average to the left and then again to replace $\eta(x+1)$ by the average to the right.

This term is the one that will change the nature of the PDE, so that for $\gamma>1$ we will derive the heat equation while for $\gamma=1$ we will derive the viscous Burgers equation.

Now we look at the boundary terms. The next terms are those on the third line of (7). From Lemma 5 we see that for $\theta,\delta<1$ (see the restriction on the parameters below) they can be replaced by

$$-\int_0^t \frac{N}{N-1}\nabla_N^+ G_s(0)r_\alpha\,ds+\int_0^t \frac{N}{N-1}\nabla_N^- G_s(1)r_\beta\,ds$$

(where r_α, r_β are defined in Section 5) while from Lemma 6 they can be replaced, for $\theta\geq 1$, by

$$-\int_0^t \frac{N}{N-1}\nabla_N^+ G_s(0)\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1)\,ds+\int_0^t \frac{N}{N-1}\nabla_N^- G_s(\tfrac{N-1}{N})\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1)\,ds.$$

The next terms on the list are those on the fourth line of (7), which, for $\theta=\delta<1$ or $\theta<1,\delta\geq 1$ and by using the fact that in this regime the space of test functions is $G\in C_0^{1,2}([0,T]\times[0,1])$, vanish as $N\to+\infty$. For $\theta>1$ and $\delta>1$ those terms vanish as $N\to+\infty$. Finally for $\theta=\delta=1$, as a consequence of Lemma 6 these terms can be replaced by

$$-\frac{N^2}{N-1}\int_0^t G_s(\tfrac{1}{N})\Big\{\frac{\kappa}{N}(\alpha-\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1))-\frac{E}{N}\alpha(1-\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1))\Big\}\,ds$$

For $\theta=1$ and $\delta>1$ those terms can be replaced by

$$-\frac{N^2}{N-1}\int_0^t G_s(\tfrac{1}{N})\Big\{\frac{\kappa}{N}(\alpha-\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1))\Big\}\,ds$$

and for $\delta=1$ and $\theta>1$ those terms can be replaced by

$$-\frac{N^2}{N-1}\int_0^t G_s(\tfrac{1}{N})\Big\{\frac{E}{N}\alpha(1-\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1))\Big\}\,ds.$$

The terms related to the right boundary can be analysed as we just did for the left boundary. Then, taking the limit $N\to\infty$ and $\varepsilon\to 0$ in in each of the expressions that we have derived, we obtain for $\gamma=1$ (for $\gamma>1$ just ignore the non linear term in the equation):

1. $\theta>1$ and $\delta>1$:

$$
\begin{aligned}
0 = {} & \langle \rho_t^N, G_t \rangle - \langle \rho_0^N, G_0 \rangle - \int_0^t \langle \rho_s, \Delta G_s \rangle \, ds - \int_0^t \langle \rho_s, \partial_s G_s \rangle \, ds \\
& - E \int_0^t \langle \nabla G_s, \rho_s(1-\rho_s) \rangle \, ds - \int_0^t \nabla G_s(0) \rho_s(0) \, ds + \int_0^t \nabla G_s(1) \rho_s(1) \, ds
\end{aligned}
$$

which corresponds to (4) with $\mathfrak{e} = E$ and $a = b = c = d = 0$.

2. $\theta = 1$ and $\delta > 1$:

$$
\begin{aligned}
0 = {} & \langle \rho_t^N, G_t \rangle - \langle \rho_0^N, G_0 \rangle - \int_0^t \langle \rho_s, \Delta G_s \rangle \, ds - \int_0^t \langle \rho_s, \partial_s G_s \rangle \, ds \\
& - E \int_0^t \langle \nabla G_s, \rho_s(1-\rho_s) \rangle \, ds - \int_0^t \nabla G_s(0) \rho_s(0) \, ds + \int_0^t \nabla G_s(1) \rho_s(1) \, ds \\
& - \int_0^t G_s(0) \kappa(\alpha - \rho_s(0)) \, ds - \int_0^t G_s(1) \kappa(\beta - \rho_s(1)) \, ds
\end{aligned}
$$

which corresponds to (4) with $\mathfrak{e} = E$ and $a = \kappa$, $b = -\kappa\alpha$, $c = -\kappa$ and $d = \kappa\beta$.

3. $\delta = 1$ and $\theta > 1$:

$$
\begin{aligned}
0 = {} & \langle \rho_t^N, G_t \rangle - \langle \rho_0^N, G_0 \rangle - \int_0^t \langle \rho_s, \Delta G_s \rangle \, ds - \int_0^t \langle \rho_s, \partial_s G_s \rangle \, ds \\
& - E \int_0^t \langle \nabla G_s, \rho_s(1-\rho_s) \rangle \, ds - \int_0^t \nabla G_s(0) \rho_s(0) \, ds + \int_0^t \nabla G_s(1) \rho_s(1) \, ds \\
& - \int_0^t G_s(0) E\alpha(1 - \rho_s(0)) \, ds + \int_0^t G_s(1) E(1-\beta) \rho_s(1) \, ds
\end{aligned}
$$

which corresponds to (4) with $\mathfrak{e} = E$ and $a = E\alpha$, $b = -E\alpha$, $c = -E(1-\beta)$ and $d = 0$.

4. $\delta = \theta = 1$:

$$
\begin{aligned}
0 = {} & \langle \rho_t^N, G_t \rangle - \langle \rho_0^N, G_0 \rangle - \int_0^t \langle \rho_s, \Delta G_s \rangle \, ds - \int_0^t \langle \rho_s, \partial_s G_s \rangle \, ds \\
& - E \int_0^t \langle \nabla G_s, \rho_s(1-\rho_s) \rangle \, ds - \int_0^t \nabla G_s(0) \rho_s(0) \, ds + \int_0^t \nabla G_s(1) \rho_s(1) \, ds \\
& - \int_0^t G_s(0)((\kappa + E)\alpha - \rho_s(0)(\kappa + E\alpha)) \, ds \\
& - \int_0^t G_s(1)(\kappa\beta - (\kappa + E(1-\beta))\rho_s(1) \, ds
\end{aligned}
$$

which corresponds to (4) with $\mathfrak{e} = E$ and $a = \kappa + E\alpha$, $b = -(\kappa + E)\alpha$, $c = -(\kappa + E(1-\beta))$ and $d = \kappa\beta$.

5. $\delta = \theta < 1$:

$$
\begin{aligned}
0 = {} & \langle \rho_t^N, G_t \rangle - \langle \rho_0^N, G_0 \rangle - \int_0^t \langle \rho_s, \Delta G_s \rangle \, ds - \int_0^t \langle \rho_s, \partial_s G_s \rangle \, ds \\
& - E \int_0^t \langle \nabla G_s, \rho_s(1-\rho_s) \rangle \, ds - \int_0^t \nabla G_s(0) \frac{\alpha(\kappa + E)}{\kappa + \alpha E} \, ds + \int_0^t \nabla G_s(1) \frac{\kappa\beta}{\kappa + E - \beta E}, ds
\end{aligned}
$$

which corresponds to (2) with $\mathfrak{e} = E$ and $a = \frac{\alpha(\kappa+E)}{\kappa+\alpha E}$ and $b = \frac{\kappa\beta}{\kappa+E-\beta E}$.

6. $\theta \in [0,1)$ and $\delta \geq 1$:

$$\begin{aligned} 0 = \langle \rho_t^N, G_t\rangle - \langle \rho_0^N, G_0\rangle - \int_0^t \langle \rho_s, \Delta G_s\rangle \, ds - \int_0^t \langle \rho_s, \partial_s G_s\rangle \, ds \\ - E\int_0^t \langle \nabla G_s, \rho_s(1-\rho_s)\rangle \, ds - \int_0^t \nabla G_s(0)\alpha \, ds + \int_0^t \nabla G_s(1)\beta \, ds \end{aligned}$$

which corresponds to (2) with $\mathfrak{e} = E$ and $a = \alpha$ and $b = \beta$.

The remaining cases of θ and δ are left open for a future work.

4 Tightness

In this section we show that the sequence of probability measures $\{\mathbb{Q}_N\}_{N\geq 1}$ is tight in the Skorohod space $\mathscr{D}([0,T],\mathscr{M}^+)$. Tightness of this sequence implies that every subsequence of $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ has a further subsequence which is weakly convergent.

Since $C^1([0,1])$ is dense in $C([0,1])$ in the uniform topology, by Proposition 1.7 of Chapter 4 in [14] it is enough to show that for every $G\in C^1([0,1])$ the sequence of measures associated with the real-valued processes $\langle \pi_t^N, G\rangle$ is tight. In order to do that, we invoke Aldous' criterion [14], stated as follows:

Lemma 1. *A sequence $\{P_N\}_{N\geq 1}$ of probability measures on $\mathscr{D}([0,T],\mathscr{M}^+)$ is tight if these two conditions hold:*

a. *For every $t\in[0,T]$ and every $\varepsilon > 0$, there exists a compact set $K_\varepsilon^t \subset \mathscr{M}^+$ such that*

$$\sup_{N\geq 1} P_N\big(\pi_t \notin K_\varepsilon^t\big) < \varepsilon$$

b. *For every $\varepsilon > 0$*

$$\lim_{l\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T, t\leq l} P_N\big(d(\pi_\tau,\pi_{\tau+t}) > \varepsilon\big) = 0,$$

where $\mathscr{T}_T$ denotes the set of stopping times with respect to the canonical filtration, bounded by T, and d is any metric in the space $\mathscr{M}^+$ inducing the weak topology.

In our case, condition **a.** reduces to

$$\lim_{A\to\infty}\sup_{N\geq 1}\mathbb{P}_{\mu_N}\big(|\langle \pi_t^N, G\rangle| > A\big) = 0. \tag{9}$$

To prove this, note that for any $A > 0$ and any $N\in\mathbb{N}$, by Markov's inequality,

$$\mathbb{P}_{\mu_N}\big(|\langle \pi_t^N, G\rangle| > A\big) \leq \frac{1}{A}\mathbb{E}_{\mu_N}\big[|\langle \pi_t^N, G\rangle|\big] = \frac{1}{A}\mathbb{E}_{\mu_N}\Big[\Big|\frac{1}{N-1}\sum_{x\in\Lambda_N} G(\tfrac{x}{N})\eta_{tN^2}(x)\Big|\Big].$$

Since $|\eta_{tN^2}(x)| \leq 1$, the last expression is bounded by $\frac{1}{A}\|G\|_\infty$, and (9) holds.

To establish condition **b.** above, we must show that, for every $\varepsilon > 0$ and $G \in C^1([0,1])$,

$$\lim_{l\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T, t\le l}\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \big|\langle\pi^N_{\tau+t},G\rangle - \langle\pi^N_\tau,G\rangle\big| > \varepsilon\Big) = 0, \tag{10}$$

where all stopping times are bounded by T (thus $\tau+t$ should be understood as $(\tau+t)\wedge T$, where $\wedge$ denotes the minimum).

Let $C^m_c([0,1])$ denote the set of all m times continuously differentiable real-valued functions with compact support contained in $(0,1)$. We begin by showing that (10) holds for functions $G \in C^2_c([0,1])$. Recall (6). Then

$$\begin{aligned}
&\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|\langle\pi^N_{\tau+t},G\rangle - \langle\pi^N_\tau,G\rangle\Big| > \varepsilon\Big)\\
=&\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|M^N_{\tau+t}(G) - M^N_\tau(G) + \int_\tau^{\tau+t} N^2\mathscr{L}_N\langle\pi^N_s,G\rangle\,ds\Big| > \varepsilon\Big)\\
\le&\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|M^N_{\tau+t}(G) - M^N_\tau(G)\Big| > \frac{\varepsilon}{2}\Big) + \mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|\int_\tau^{\tau+t} N^2\mathscr{L}_N\langle\pi^N_s,G\rangle\,ds\Big| > \frac{\varepsilon}{2}\Big).
\end{aligned}$$

Applying, to the first term on the RHS of last inequality, Chebyshev's (resp. Markov's) inequality to the first (resp. second), we can bound the previous expression by

$$\frac{4}{\varepsilon^2}\mathbb{E}_{\mu_N}\Big[\Big(M^N_{\tau+t}(G) - M^N_\tau(G)\Big)^2\Big] + \frac{2}{\varepsilon}\mathbb{E}_{\mu_N}\Big[\Big|\int_\tau^{\tau+t} N^2\mathscr{L}_N\langle\pi^N_s,G\rangle\,ds\Big|\Big].$$

Therefore it is enough to show the next two limits:

$$\lim_{l\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T, t\le l}\mathbb{E}_{\mu_N}\Big[\Big|\int_\tau^{\tau+t} N^2\mathscr{L}_N\langle\pi^N_s,G\rangle\,ds\Big|\Big] = 0 \tag{11}$$

$$\lim_{l\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T, t\le l}\mathbb{E}_{\mu_N}\Big[\Big(M^N_{\tau+t}(G) - M^N_\tau(G)\Big)^2\Big] = 0. \tag{12}$$

Since we are assuming that $G \in C^2_c([0,1])$, there exists $N_0 \in \mathbb{N}$ such that $G(0) = G(\frac{1}{N}) = G(\frac{N-1}{N}) = G(1) = 0$ for all $N \ge N_0$. Since $|\eta_{sN^2}(x)| \le 1$, we have

$$\begin{aligned}
\Big|N^2\mathscr{L}_N\langle\pi^N_s,G\rangle\Big| &= \Big|\langle\pi^N_s,\Delta_N G\rangle + \frac{E}{N-1}\sum_{x=1}^{N-2}\nabla^+_N G(\tfrac{x}{N})\eta_{sN^2}(x)(1-\eta_{sN^2}(x+1))\Big|\\
&\le \|G''\|_\infty + E\|G'\|_\infty
\end{aligned}$$

for $N \ge N_0$. Therefore there exists a constant C such that $|N^2\mathscr{L}_N\langle\pi^N_s,G\rangle| \le C$ for all $N \in \mathbb{N}$, and (11) follows.

Let us now prove (12). By Dynkin's formula, see for example Lemma 5.1 of [14],

$$M_t^N(G)^2 - \int_0^t N^2[\mathscr{L}_N \langle \pi_s^N, G\rangle^2 - 2\langle \pi_s^N, G\rangle \mathscr{L}_N \langle \pi_s^N, G\rangle]\, ds$$

is a martingale with respect to $\{\mathscr{F}_t\}_{t\geq 0}$. From this, and since τ is a stopping time, it follows that

$$\mathbb{E}_{\mu_N}\Big[\Big(M_{\tau+t}^N(G) - M_\tau^N(G)\Big)^2\Big] = \mathbb{E}_{\mu_N}\Big[\int_\tau^{\tau+t} N^2[\mathscr{L}_N \langle \pi_s^N, G\rangle^2 - 2\langle \pi_s^N, G\rangle \mathscr{L}_N \langle \pi_s^N, G\rangle] ds\Big].$$

A computation shows that, for $N \geq N_0$,

$$\begin{aligned} &N^2[\mathscr{L}_N \langle \pi_s^N, G\rangle^2 - 2\langle \pi_s^N, G\rangle \mathscr{L}_N \langle \pi_s^N, G\rangle] \\ &= \frac{1}{(N-1)^2} \sum_{x=1}^{N-2} c_{x,x+1}(\eta_{sN^2})(\eta_{sN^2}(x) - \eta_{sN^2}(x+1))^2 (\nabla_N^+ G(\tfrac{x}{N}))^2. \end{aligned}$$

Since $|c_{x,x+1}(\eta_{sN^2})| \leq 2 + E$, the last expression is bounded by $\frac{2+E}{N}\|G'\|_\infty$, and therefore it goes to zero as $N \to \infty$. This finishes the proof for $G \in C_c^2([0,1])$.

Assume now that $G \in C^1([0,1])$. Since $C_c^2([0,1])$ is dense in $C^1([0,1])$wrt the $\mathbb{L}^1$ topology, there exists a sequence $\{G_k\}_{k\geq 1}$ of functions in $C_c^2([0,1])$ such that $\|G_k - G\|_1 \to 0$. Since the probability in (10) is less than or equal to

$$\mathbb{P}_{\mu_N}\left(\eta_\cdot : \left|\langle \pi_{\tau+t}^N, G_k\rangle - \langle \pi_\tau^N, G_k\rangle\right| > \frac{\varepsilon}{2}\right) + \mathbb{P}_{\mu_N}\left(\eta_\cdot : \left|\langle \pi_{\tau+t}^N, G - G_k\rangle - \langle \pi_\tau^N, G - G_k\rangle\right| > \frac{\varepsilon}{2}\right)$$

and $G_k \in C_c^2([0,1])$, by the computation above it remains only to check that the last probability vanishes as $N \to \infty$ and then $k \to \infty$. For that purpose, note that

$$\begin{aligned} \big|\langle \pi_{\tau+t}^N, G - G_k\rangle - \langle \pi_\tau^N, G - G_k\rangle\big| &\leq \frac{2}{N} \sum_{x=1}^{N-1} \big|(G - G_k)(\tfrac{x}{N})\big| \\ &\leq 2 \sum_{x=1}^{N-1} \int_{\frac{x}{N}}^{\frac{x+1}{N}} \big|(G - G_k)(\tfrac{x}{N}) - (G - G_k)(q)\, dq\big| + 2\int_0^1 \big|(G - G_k)(q)\big|\, dq \\ &\leq \frac{2}{N}\|(G - G_k)'\|_\infty + 2\int_0^1 \big|(G - G_k)(q)\big|\, dq. \end{aligned}$$

The result follows by taking $N \to \infty$ and then $k \to \infty$.

5 Replacement lemmas

This section is devoted to the proof of the two lemmas mentioned in Section 3, using methods similar to those in [2] and [10].

5.1 Estimates on Dirichlet forms

We consider $\rho : [0,1] \to [0,1]$ to be a Lipschitz continuous function and we take $r_\alpha, r_\beta \in (0,1)$ satisfying $r_\alpha \leq \rho(q) \leq r_\beta$ for $q \in [0,1]$ such that $\rho(0) = r_\alpha$, $\rho(1) = r_\beta$. Denote by $\nu^N_{\rho(\cdot)}$ the Bernoulli product measure on Ω_N with marginals

$$\nu^N_{\rho(\cdot)}\{\eta : \eta(x) = 1\} = \rho(\tfrac{x}{N}),$$

for $x \in \Lambda_N$. For functions $f, g : \Omega_N \to \mathbb{R}$ and a probability measure μ on Ω_N, the inner product in $L^2(\Omega_N, \mu)$ is denoted by $\langle f, g\rangle_\mu = \int_{\Omega_N} f(\eta) g(\eta) d\mu$.

Lemma 2. *Let $T : \Omega_N \to \Omega_N$ be a transformation, (either the exchange $\eta^{x,y}$ or the flip η^x), $c : \Omega_N \to \mathbb{R}$ a positive function and f a density with respect to a probability measure μ on Ω_N. Then*

$$\begin{aligned}\int c(\eta)\left[\sqrt{f(T(\eta))} - \sqrt{f(\eta)}\right]\sqrt{f(\eta)}\, d\mu \\ = -\frac{1}{2}\int c(\eta)\left[\sqrt{f(T(\eta))} - \sqrt{f(\eta)}\right]^2 d\mu \\ + \frac{1}{2}\int \left[\sqrt{f(T(\eta))}\right]^2 \left[c(\eta) - c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\right] d\mu.\end{aligned} \tag{13}$$

Proof. To prove the result, it is enough to write the term at the LHS of (13) as its half plus its half and to sum and subtract the term needed to complete the square.

We recall Lemmas 5.1 and 5.2 of [2]:

Lemma 3. *Let $T : \Omega_N \to \Omega_N$ be a transformation, $c : \Omega_N \to \mathbb{R}$ a positive function and f a density with respect to a probability measure μ on Ω_N. Then*

$$\begin{aligned}&\left\langle c(\eta)\big(\sqrt{f(T(\eta))} - \sqrt{f(\eta)}\big), \sqrt{f(\eta)}\right\rangle_\mu \\ &\leq -\frac{1}{4}\int_{\Omega_N} c(\eta)\Big(\sqrt{f(T(\eta))} - \sqrt{f(\eta)}\Big)^2 d\mu \\ &+ \frac{1}{16}\int_{\Omega_N} \frac{1}{c(\eta)}\left[c(\eta) - c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\right]^2 \Big(\sqrt{f(T(\eta))} + \sqrt{f(\eta)}\Big)^2 d\mu\end{aligned}$$

Lemma 4. *Let $\rho(\cdot)$ be as described above. There exists a constant C such that, for any $N \in \mathbb{N}$ and any density f with respect to $\nu^N_{\rho(\cdot)}$,*

$$\sup_{1 \leq x \leq N-2} \int_{\Omega_N} f(\eta^{x,x+1}) d\nu^N_{\rho(\cdot)}(\eta) \leq C, \quad \sup_{x \in \{1, N-1\}} \int_{\Omega_N} f(\eta^x) d\nu^N_{\rho(\cdot)}(\eta) \leq C.$$

We introduce the following non-negative functions, defined for densities f with respect to $\nu^N_{\rho(\cdot)}$:

$$\begin{aligned}
&\mathscr{D}_N^L(\sqrt{f},\nu^N_{\rho(\cdot)}) = \int_{\Omega_N} c_{0,1}(\eta)\big(\sqrt{f(\eta^1)}-\sqrt{f(\eta)}\big)^2 d\nu^N_{\rho(\cdot)},\\
&\mathscr{D}_N^R(\sqrt{f},\nu^N_{\rho(\cdot)}) = \int_{\Omega_N} c_{N-1,N}(\eta)\big(\sqrt{f(\eta^{N-1})}-\sqrt{f(\eta)}\big)^2 d\nu^N_{\rho(\cdot)},\\
&\mathscr{D}_N^B(\sqrt{f},\nu^N_{\rho(\cdot)}) = \sum_{x=1}^{N-2}\int_{\Omega_N} c_{x,x+1}(\eta)\big(\sqrt{f(\eta^{x,x+1})}-\sqrt{f(\eta)}\big)^2 d\nu^N_{\rho(\cdot)}.
\end{aligned}$$

As a consequence of Lemma 2 we conclude that

Corollary 1. *Let $\rho(\cdot)$ be a profile. There exists a constant $C>0$ such that, for any density f and $N\in\mathbb{N}$,*

$$\langle \mathscr{L}_N^L\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} \leq -\frac{1}{2}\mathscr{D}_N^L(\sqrt{f},\nu^N_{\rho(\cdot)}) + C\Big|\frac{\kappa}{N^\theta}\Big(\rho(\tfrac{1}{N})-\alpha\Big)+\frac{\alpha E}{N^\delta}\Big(\rho(\tfrac{1}{N})-1\Big)\Big|, \tag{14}$$

$$\begin{aligned}
\langle \mathscr{L}_N^R\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} \leq& -\frac{1}{2}\mathscr{D}_N^R(\sqrt{f},\nu^N_{\rho(\cdot)})\\
&+C\Big|\frac{\kappa}{N^\theta}\Big(\rho(\tfrac{N-1}{N})-\beta\Big)+\frac{(1-\beta)E}{N^\delta}\rho(\tfrac{N-1}{N})\Big|.
\end{aligned} \tag{15}$$

Proof. We present the proof for the left boundary since the other case is analogous. By Lemma 2,

$$\begin{aligned}
\langle \mathscr{L}_N^L\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} =& -\frac{1}{2}\mathscr{D}_N^L(\sqrt{f},\nu^N_{\rho(\cdot)})\\
&+\frac{1}{2}\int_{\Omega_N}\Big[c_{0,1}(\eta)-c_{0,1}(\eta^1)\frac{\nu^N_{\rho(\cdot)}(\eta^1)}{\nu^N_{\rho(\cdot)}(\eta)}\Big]\Big(\sqrt{f(\eta^1)}\Big)^2 d\nu^N_{\rho(\cdot)}.
\end{aligned}$$

Let $\bar\eta\in\{0,1\}^{\{2,\dots,N-1\}}$ denote the configuration obtained from η by discarding its value at 1, so that $\eta=(1,\bar\eta)$ if $\eta(1)=1$ and $\eta=(0,\bar\eta)$ if $\eta(1)=0$. The second term on the RHS of last expression inside can be written as

$$\begin{aligned}
&\frac{1}{2}\sum_{\bar\eta}\Big[\frac{(1-\alpha)\kappa}{N^\theta}-\alpha\Big(\frac{\kappa}{N^\theta}+\frac{E}{N^\delta}\Big)\frac{(1-\rho(\frac{1}{N}))}{\rho(\frac{1}{N}))}\Big] f(0,\bar\eta)\nu^N_{\rho(\cdot)}(1,\bar\eta)\\
+&\frac{1}{2}\sum_{\bar\eta}\Big[\alpha\Big(\frac{\kappa}{N^\theta}+\frac{E}{N^\delta}\Big)-\frac{(1-\alpha)\kappa}{N^\theta}\frac{\rho(\frac{1}{N})}{(1-\rho(\frac{1}{N}))}\Big] f(1,\bar\eta)\nu^N_{\rho(\cdot)}(0,\bar\eta)\\
\leq& C\Big|\frac{\kappa}{N^\theta}\Big(\rho(\tfrac{1}{N})-\alpha\Big)+\frac{\alpha E}{N^\delta}\Big(\rho(\tfrac{1}{N})-1\Big)\Big|.
\end{aligned}$$

Corollary 2. *Let $\rho(\cdot)$ be a Lipschitz continuous function. There exists a constant $C>0$ such that, for any density f and $N\in\mathbb{N}$,*

$$\langle \mathscr{L}_N^B\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} \leq -\frac{1}{4}\mathscr{D}_N^B(\sqrt{f},\nu^N_{\rho(\cdot)}) + C\Big(\frac{1}{N}+\frac{1}{N^{2\gamma-1}}\Big)$$

Proof. By Lemma 3,

$$\begin{aligned}\langle \mathscr{L}_N^B\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} &\leq -\frac{1}{4}\mathscr{D}_N^B(\sqrt{f},\nu^N_{\rho(\cdot)})\\ &+\frac{1}{16}\sum_{x=1}^{N-2}\int_{\Omega_N}\frac{1}{c_{x,x+1}(\eta)}\Big[c_{x,x+1}(\eta)-c_{x,x+1}(\eta^{x,x+1})\frac{\nu^N_{\rho(\cdot)}(\eta^{x,x+1})}{\nu^N_{\rho(\cdot)}(\eta)}\Big]^2\times\\ &\times\Big(\sqrt{f(\eta^{x,x+1})}+\sqrt{f(\eta)}\Big)^2 d\nu^N_{\rho(\cdot)}.\end{aligned}$$

For any $1\leq x\leq N-2$, the expression inside last sum is equal to

$$\sum_{\zeta}\frac{1}{1+\frac{E}{N^\gamma}}\left[1+\frac{E}{N^\gamma}-\frac{\rho(\frac{x+1}{N})(1-\rho(\frac{x}{N}))}{(1-\rho(\frac{x+1}{N}))\rho(\frac{x}{N})}\right]^2\Big(\sqrt{f(\zeta)}+\sqrt{f(\zeta^{x,x+1})}\Big)^2\nu^N_{\rho(\cdot)}(\zeta)$$

$$+\sum_{\zeta}\left[1-\left(1+\frac{E}{N^\gamma}\right)\frac{\rho(\frac{x}{N})(1-\rho(\frac{x+1}{N}))}{(1-\rho(\frac{x}{N}))\rho(\frac{x+1}{N})}\right]^2\Big(\sqrt{f(\zeta)}+\sqrt{f(\zeta^{x,x+1})}\Big)^2\nu^N_{\rho(\cdot)}(\zeta)$$

where the first sum is over all configurations $\zeta\in\Omega_N$ satisfying $\zeta(x)=1$, $\zeta(x+1)=0$, and the second sum over those $\zeta\in\Omega_N$ such that $\zeta(x)=0$, $\zeta(x+1)=1$. By Lemma 4, we conclude that

$$\langle \mathscr{L}_N^B\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} \leq -\frac{1}{4}\mathscr{D}_N^B(\sqrt{f},\nu^N_{\rho(\cdot)})+C\sum_{x=1}^{N-2}\left(\big|\rho(\tfrac{x+1}{N})-\rho(\tfrac{x}{N})\big|+\frac{E}{N^\gamma}\right)^2$$

for some constant C. Since $\rho(\cdot)$ is Lipschitz continuous, we get

$$\langle \mathscr{L}_N^B\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} \leq -\frac{1}{4}\mathscr{D}_N^B(\sqrt{f},\nu^N_{\rho(\cdot)})+C\Big(\frac{1}{N}+\frac{1}{N^{2\gamma-1}}\Big).$$

Remark 2. Now, we analyze the bounds obtained above in different regimes of θ and δ. Observe that the bound above simplifies to (without requiring anything on the values of the profile $\rho(\cdot)$ at $q=0$ nor at $q=1$)

$$\langle \mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} \leq -\frac{1}{4}\mathscr{D}_N(\sqrt{f},\nu^N_{\rho(\cdot)})+C\Big(\frac{1}{N^\theta}+\frac{1}{N}+\frac{1}{N^\delta}+\frac{1}{N^{2\gamma-1}}\Big). \tag{16}$$

This will be useful in the cases $\theta\geq 1$ and $\delta\geq 1$ below. When $\theta=\delta<1$, we ask that $\rho(0)=r_\alpha$ and $\rho(1)=r_\beta$, where r_α and r_β are defined as

$$r_\alpha=\frac{\alpha(\kappa+E)}{\kappa+\alpha E},\ \ r_\beta=\frac{\kappa\beta}{\kappa+E-\beta E}, \tag{17}$$

and, since we assumed $\rho(\cdot)$ to be Lipschitz, we get the bound:

$$\langle \mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} \leq -\frac{1}{4}\mathscr{D}_N(\sqrt{f},\nu^N_{\rho(\cdot)})+C\Big(\frac{1}{N^{\theta+1}}+\frac{1}{N}+\frac{1}{N^{2\gamma-1}}\Big). \tag{18}$$

In the case $\delta \geq 1$ and $\theta \in [0,1)$, by asking that $\rho(0)=\alpha$ (and $\rho(1)=\beta$) and $\rho(\cdot)$ to be Lipschitz we get the bound:

$$\langle \mathscr{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu^N_{\rho(\cdot)}} \leq -\frac{1}{4}\mathscr{D}_N(\sqrt{f}, \nu^N_{\rho(\cdot)}) + C\Big(\frac{1}{N^{\theta+1}} + \frac{1}{N^{\delta}} + \frac{1}{N} + \frac{1}{N^{2\gamma-1}}\Big). \quad (19)$$

Now that we have all the ingredients we need we state and prove the replacement lemmas we need. The next lemma will be used in the Dirichlet case.

Lemma 5. *Let $F : \mathbb{N} \times [0,T] \to \mathbb{R}$ be a bounded function and suppose that*

(1) $\theta = \delta < 1$
(2) $\theta \in [0,1)$ *and* $\delta \geq 1$.

Then, for any $t \in [0,T]$,

$$\lim_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t F(N,s)(\eta_{sN^2}(1) - r_\alpha)\, ds\right|\right] = 0,$$

where in

(1) r_α *and* r_β *where defined in* (17).
(2) $r_\alpha = \alpha$ *and* $r_\beta = \beta$.

The same estimate above holds replacing $\eta(1)$ by $\eta(N-1)$ and r_α by r_β.

The next lemma will be used in the Robin cases for $x=1$ and $x=N-1$ to deal with the boundary terms; and in all the cases for $x \in \Lambda_n^\varepsilon$ to treat the bulk term coming from the weak asymmetry.

Lemma 6. *Fix x. Let $F : \mathbb{N} \times \Omega_N \times [0,T] \to \mathbb{R}$ be a bounded function whose support does not intersect the set of points $\{x+1, \cdots, x+\varepsilon N\}$. Then, for any $t \in [0,T]$ and for any x*

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t F(N,\eta_{sN^2},s)(\eta_{sN^2}(x) - \overrightarrow{\eta}^{\varepsilon N}_{sN^2}(x))\, ds\right|\right] = 0$$

where $\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(x)$ was defined in (8). *The same result holds replacing $\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(x)$ by $\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(x)$, but in this case the support of F cannot intersect the set of points $\{x-\varepsilon N+1, \cdots, x\}$. The values above of x are restricted to the cases when the averages make sense.*

In order to prove Lemmas 5 and 6, we need some intermediate results. First observe that if $H(\mu_N|\nu^N_{\rho(\cdot)})$ is the relative entropy of the measure μ_N with respect to $\nu^N_{\rho(\cdot)}$, then there exists a constant $C_{\alpha,\beta}$ such that $H(\mu_N|\nu^N_{\rho(\cdot)}) \leq C_{\alpha,\beta}N$. To see this it is enough to observe that

$$\nu^N_{\rho(\cdot)}(\eta) = \prod_{x=1}^{N-1} \rho(\tfrac{x}{N})^{\eta(x)}(1-\rho(\tfrac{x}{N}))^{1-\eta(x)} \geq (r_\alpha \wedge (1-r_\beta))^N := (C_{\alpha,\beta})^N,$$

and by the explicit formula for the entropy, it holds

$$H(\mu_N|\nu^N_{\rho(\cdot)}) = \sum_{\eta\in\Omega_N} \mu_N(\eta)\log\left(\frac{\mu_N(\eta)}{\nu^N_{\rho(\cdot)}(\eta)}\right) \leq N\log\left(\frac{1}{C_{\alpha,\beta}}\right)\sum_{\eta\in\Omega_N}\mu_N(\eta) = C_{\alpha,\beta}N.$$

Lemma 7. *Let $G_N : [0,T]\times\Omega_N \to \mathbb{R}$, $N\in\mathbb{N}$, be a sequence of functions and $t\in[0,T]$. Then, for any $N\in\mathbb{N}$ and for any $B>0$,*

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t G_N(s,\eta_{sN^2})\,ds\right|\right] \leq \frac{C_{\alpha,\beta}}{B} + \int_0^t \sup_f\left\{\langle G_N(s,\eta),f\rangle_{\nu^N_{\rho(\cdot)}} + \frac{N}{B}\langle\mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}}\right\}ds,$$

where the supremum is taken over all densities f with respect to $\nu^N_{\rho(\cdot)}$, and $C_{\alpha,\beta}$ is a positive constant.

Proof. By the entropy inequality and Jensen's inequality, for any $B>0$ the expectation in the statement of the theorem is bounded by

$$\frac{C_0}{B} + \frac{1}{NB}\log\mathbb{E}_{\nu^N_{\rho(\cdot)}}\left[e^{|\int_0^t BNG_N(s,\eta_{sN^2})\,ds|}\right]. \tag{20}$$

Since $e^{|x|}\leq e^x+e^{-x}$ and

$$\limsup_{N\to\infty}\tfrac{1}{N}\log(a_N+b_N) \leq \max\left\{\limsup_{N\to\infty}\tfrac{1}{N}\log(a_N), \limsup_{N\to\infty}\tfrac{1}{N}\log(b_N)\right\},$$

we can remove the absolute value from expression (20). By Feynman-Kac's formula (see Lemma 7.3 in [1]), (20) is bounded by

$$\frac{C_{\alpha,\beta}}{B} + \int_0^t \sup_f\left\{\langle G_N(s,\eta),f\rangle_{\nu^N_{\rho(\cdot)}} + \frac{N}{B}\langle\mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}}\right\}ds,$$

with the supremum taken over all densities f with respect to $\nu^N_{\rho(\cdot)}$.

In the next lemma we do not assume any condition on the profile $\rho(\cdot)$.

Lemma 8. *There exists a positive constant C such that, for any $A>0$, any $N\in\mathbb{N}$ and any density f with respect to $\nu^N_{\rho(\cdot)}$,*

$$\left|\langle\eta(1)-r_\alpha,f\rangle_{\nu^N_{\rho(\cdot)}}\right| \leq \frac{C}{A}\mathscr{D}^L_N(\sqrt{f},\nu^N_{\rho(\cdot)}) + CAN^\theta + C\left|\rho(\tfrac{1}{N})-r_\alpha\right|,$$

Proof. By summing and subtracting the appropriate term,

$$\begin{aligned}\big|\langle \eta(1)-r_\alpha, f\rangle_{\nu^N_{\rho(\cdot)}}\big| \leq \frac{1}{2}\Big|\int_{\Omega_N}(\eta(1)-r_\alpha)(f(\eta)-f(\eta^1))d\nu^N_{\rho(\cdot)}\Big| \\ +\frac{1}{2}\Big|\int_{\Omega_N}(\eta(1)-r_\alpha)(f(\eta)+f(\eta^1))d\nu^N_{\rho(\cdot)}\Big|.\end{aligned} \tag{21}$$

By multiplying and dividing by $\sqrt{c_{0,1}(\eta)}$ and applying Young's inequality we have, for any $A>0$, that the first term on the RHS of (21) can be bounded by

$$\begin{aligned}&\frac{A}{4}\int_{\Omega_N}\frac{(\eta(1)-r_\alpha)^2}{c_{0,1}(\eta)}\big(\sqrt{f(\eta)}+\sqrt{f(\eta^1)}\big)^2 d\nu^N_{\rho(\cdot)}+\frac{1}{4A}\mathscr{D}^L_N(\sqrt{f},\nu^N_{\rho(\cdot)})\\ \leq& C'AN^\theta+\frac{1}{A}\mathscr{D}^L_N(\sqrt{f},\nu^N_{\rho(\cdot)}),\end{aligned}$$

where the last inequality holds, for some constant C', by Lemma 4 and by the fact that $(\eta(1)-r_\alpha)^2 c_{0,1}(\eta)^{-1}$ is bounded by a constant times N^θ.

Now we analyze the second term on the RHS of (21). Let $\bar\eta\in\{0,1\}^{\{2,\dots,N-1\}}$ denote the configuration obtained from η by discarding its value at 1, so that $\eta=(1,\bar\eta)$ if $\eta(1)=1$ and $\eta=(0,\bar\eta)$ if $\eta(1)=0$. Since $\nu^N_{\rho(\cdot)}$ is a product measure with $\nu^N_{\rho(\cdot)}(\eta(1)=1)=\rho(\frac{1}{N})$, the second term on the RHS of (21) is equal to

$$\begin{aligned}&\frac{1}{2}\Big|\sum_{\bar\eta}\Big[(1-r_\alpha)\rho(\tfrac{1}{N})-r_\alpha(1-\rho(\tfrac{1}{N}))\Big](f(0,\bar\eta)+f(1,\bar\eta))\nu^N_{\rho(\cdot)}(\bar\eta)\Big|\\ &=\frac{1}{2}\Big|\sum_{\bar\eta}(\rho(\tfrac{1}{N})-r_\alpha)(f(1,\bar\eta)+f(0,\bar\eta))\nu^N_{\rho(\cdot)}(\bar\eta)\Big|\\ &\leq C''|\rho(\tfrac{1}{N})-r_\alpha|\sum_{\bar\eta}\big[\rho(\tfrac{1}{N})f(1,\bar\eta)\nu^N_{\rho(\cdot)}(\bar\eta)+(1-\rho(\tfrac{1}{N}))f(0,\bar\eta)\nu^N_{\rho(\cdot)}(\bar\eta)\big]\\ &=C''|\rho(\tfrac{1}{N})-r_\alpha|\sum_{\eta}f(\eta)\nu^N_{\rho(\cdot)}(\eta)=C''|\rho(\tfrac{1}{N})-r_\alpha|,\end{aligned}$$

where $C''=\max\left\{\frac{1}{2r_\alpha},\frac{1}{2(1-r_\beta)}\right\}\geq\max\left\{\frac{1}{2\rho(\frac{1}{N})},\frac{1}{2(1-\rho(\frac{1}{N}))}\right\}$.

Now we have all set to prove the lemma. We start with the proof of Lemma 5.

5.2 *Proof of Lemma 5*

The main difference in the proof between the two cases (1) and (2) is that we change from μ_N to a measure $\nu^N_{\rho(\cdot)}$ where the profile $\rho:[0,1]\to[0,1]$ is Lipschitz, but we assume $\rho(0)=r_\alpha$ and $\rho(1)=r_\beta$ where in (1) (resp. (2)) these values are given in (17) (resp. $r_\alpha=\alpha$ and $r_\beta=\beta$). By Lemma 7, since $|F(N,s)|\leq C_1$ for some constant $C_1>0$, the expectation in the statement of the lemma is bounded from above by

$$\frac{C_{\alpha,\beta}}{B}+t\sup_{f}\Big\{C_1\big|\langle\eta(1)-r_\alpha,f\rangle_{\nu^N_{\rho(\cdot)}}\big|+\frac{N}{B}\langle\mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}}\Big\}. \tag{22}$$

If $\theta=\delta<1$, we recall (18) and the term inside the supremum in the last expression is bounded from above by

$$C_1\big|\langle\eta(1)-r_\alpha,f\rangle_{\nu^N_{\rho(\cdot)}}\big|-\frac{N}{4B}\mathscr{D}_N(\sqrt{f},\nu^N_{\rho(\cdot)})+\frac{C}{B}\Big(1+\frac{1}{N^\theta}\Big).$$

If $\theta\in[0,1)$ and $\delta\geq 1$, we recall (19) and the term inside the supremum in (22) is bounded from above by

$$C_1\big|\langle\eta(1)-r_\alpha,f\rangle_{\nu^N_{\rho(\cdot)}}\big|-\frac{N}{4B}\mathscr{D}_N(\sqrt{f},\nu^N_{\rho(\cdot)})+\frac{C}{B}\Big(1+\frac{1}{N^\theta}\Big).$$

Furthermore, from Lemma 8, and choosing $A=cBN^{-1}$ where c is a suitable constant, the last bounds obtained above for both cases give that the term in the supremum in (22) is bounded from above by a constant times

$$\frac{BN^\theta}{N}+\big|\rho(\tfrac{1}{N})-r_\alpha\big|+\frac{C}{B}\Big(1+\frac{1}{N^\theta}\Big).$$

Since we assumed $\rho(0)=r_\alpha$ (on the case $\theta=\delta$) or $\rho(0)=\alpha$ (on the case $\theta\in[0,1)$ and $\delta\geq 1$) and that $\rho(\cdot)$ is Lipschitz continuous, all the terms above depending on N vanish as $N\to\infty$. To finish the proof we just need to send $B\to+\infty$.

5.3 *Proof of Lemma 6*

This result is necessary in all the regimes of θ and δ. The main difference in the proof between these cases is the fact that we change from μ_N to a measure $\nu^N_{\rho(\cdot)}$, where the profile $\rho:[0,1]\to[0,1]$ is Lipschitz continuous, but in the case $\delta>1$ and $\theta>1$ we do not need to impose any extra restriction on the profile. In the other cases we assume the extra conditions as in the proof of last lemma. Using the same arguments as above, the expectation in the statement of the lemma is bounded from above by

$$\frac{C_{\alpha,\beta}}{B}+\int_0^t\sup_{f}\Big\{\big|\langle F(N,\eta,s)(\eta(x)-\overrightarrow{\eta}^{\varepsilon N}(x)),f\rangle_{\nu^N_\rho}\big|+\frac{N}{B}\langle\mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}}\Big\}ds, \tag{23}$$

for any $B>0$. By writing the term above as a telescopic sum, summing and subtracting terms, we write

$$
\begin{aligned}
&\langle F(N,\eta,s)(\eta(x)-\overrightarrow{\eta}^{\varepsilon N}(x)),f\rangle_{\nu^N_{\rho(\cdot)}}\\
&\quad=\frac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\int F(N,\eta,s)(\eta(z)-\eta(z+1))\Big(f(\eta)-f(\eta^{z,z+1})\Big)d\nu^N_{\rho(\cdot)}\\
&\quad+\frac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\int F(N,\eta,s)(\eta(z)-\eta(z+1))\Big(f(\eta)+f(\eta^{z,z+1})\Big)d\nu^N_{\rho(\cdot)}.
\end{aligned}
$$

By Young's inequality, the first term on the RHS of last expression can be bounded from above by a constant times

$$
\begin{aligned}
&\frac{\|F\|_\infty^2 A}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\int c_{z,z+1}(\eta)(\sqrt{f(\eta^{z,z+1})}-\sqrt{f(\eta)})^2 d\nu^N_{\rho(\cdot)}\\
&+\frac{\|F\|_\infty^2}{A\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\int \frac{1}{c_{z,z+1}(\eta)}(\sqrt{f(\eta^{z,z+1})}+\sqrt{f(\eta)})^2(\eta(z+1)-\eta(z))^2 d\nu^N_{\rho(\cdot)}
\end{aligned}
$$

for any $A>0$. The first term in last expression can be bounded by $A\|F\|_\infty^2\mathscr{D}_N^B(\sqrt{f},\nu^N_{\rho(\cdot)})$ and by Lemma 4, the second term can be bounded by a constant times $A^{-1}\|F\|_\infty^2\varepsilon N$.

The remaining term in last display can be treated using a similar argument to the one as in the last part of the proof of Lemma 8, invoking the fact that the profile is Lipschitz, and can be bounded from above by ε.

Now we split the proof according to the regimes of δ and θ. For simplicity of the presentation we restrict to the case $\theta\geq 1$ and $\delta\geq 1$. The other cases are left to the reader. From (16) we get

$$
\langle \mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_\rho}\leq -\frac{1}{4}\mathscr{D}_N(\sqrt{f},\nu^N_{\rho(\cdot)})+\frac{C}{N}.
$$

Then (23) becomes bounded from above by

$$
\frac{C_{\alpha,\beta}}{B}+\int_0^t\sup_f\Big\{A\|F\|_\infty^2\mathscr{D}_N^B(\sqrt{f},\nu^N_{\rho(\cdot)})+\frac{\|F\|_\infty^2\varepsilon N}{A}+\varepsilon-\frac{N}{4B}\mathscr{D}_N(\sqrt{f},\nu^N_{\rho(\cdot)})+\frac{C}{4B}\Big\}ds,
$$

Therefore, choosing $A=N(C_1B)^{-1}$, where C_1 is an appropriate constant, the expression in last supremum is bounded by a constant times

$$
B\varepsilon+\frac{1}{B}+\varepsilon.
$$

The result follows by taking the limit $\varepsilon\to 0$ and then $B\to\infty$.

Appendix: Uniqueness of weak solutions

In this section we prove uniqueness of weak solutions of (2). We use the method taken from Appendix 2 of [14] but an alternative proof can be seen in, for example, Section 7 of [9]. Take ρ^1 and ρ^2 two weak solutions of (2) with the same initial condition and call $\bar{\rho}$ their difference $\bar{\rho} = \rho^1 - \rho^2$. Let us recall that the set $\{\psi_k\}_k$ of eigenfunctions of the Laplacian with Dirichlet boundary conditions, given by $\psi_k(u) = \sqrt{2}\sin(k\pi u)$ for $k \geq 1$ and $\psi_0(u) = 1$ is an orthonormal basis of $L^2([0,1])$. Consider

$$V(t) = \sum_{k\geq 0} \frac{1}{2a_k} < \bar{\rho}_t, \psi_k >^2$$

where $a_k = (k\pi)^2 + 1$. We claim that $V'(t) \leq CV(t)$, where C is a positive constant and since $V(0) = 0$, from Gronwall's inequality we conclude that $V(t) \leq 0$, and we are done. Now,

$$V'(t) = \sum_{k\geq 0} \frac{1}{a_k} < \bar{\rho}_t, \psi_k > \frac{d}{dt} < \bar{\rho}_t, \psi_k > . \tag{24}$$

and from (3), $\frac{d}{dt} < \bar{\rho}_t, \psi_k >$ can be replaced by

$$< \bar{\rho}_t, \Delta\psi_k > + \mathfrak{c} < \sigma(\rho^1_t) - \sigma(\rho^2_t), \partial_u \psi_k > = -(k\pi)^2 < \bar{\rho}_t, \psi_k > + \mathfrak{c} < \sigma(\rho^1_t) - \sigma(\rho^2_t), \partial_u \psi_k >$$

and we get

$$V'(t) = -\sum_{k\geq 0} \frac{(k\pi)^2}{a_k} < \bar{\rho}_t, \psi_k >^2 + \sum_{k\geq 0} \frac{\mathfrak{c}}{a_k} < \bar{\rho}_t, \psi_k > < \sigma(\rho^1_t) - \sigma(\rho^2_t), \partial_u \psi_k > .$$

From Young's inequality the rightmost term in last display is bounded from above by

$$\frac{1}{2A} \sum_{k\geq 0} \frac{\mathfrak{c}}{a_k} < \bar{\rho}_t, \psi_k >^2 + \frac{A}{2} \sum_{k\geq 0} \frac{\mathfrak{c}}{a_k} < \sigma(\rho^1_t) - \sigma(\rho^2_t), \partial_u \psi_k >^2,$$

for any $A > 0$. Observe that $\partial_u \psi_k(u) = -k\pi\, \varphi_k(u)$, with $\varphi_k(u) = \sqrt{2}\cos(k\pi u)$, for $k \geq 1$ and $\varphi_0(u) = 1$. Therefore, the rightmost term in last display can be bounded from above by

$$\frac{A}{2} \sum_{k\geq 0} \frac{\mathfrak{c}(k\pi)^2}{a_k} < \sigma(\rho^1_t) - \sigma(\rho^2_t), \varphi_k >^2 \leq \frac{A}{2} \mathfrak{c} \sum_{k\geq 0} < \sigma(\rho^1_t) - \sigma(\rho^2_t), \varphi_k >^2,$$

because of the choice of a_k. Observe that, since $\{\varphi_k\}_k$ is an orthonormal basis of $\mathbb{L}^2$, the rightmost term in last display is equal to $\frac{A}{2}\mathfrak{c}\int_0^1 \big(\sigma(\rho^1_t) - \sigma(\rho^2_t)\big)^2 du$. Since $(\sigma(\rho^1_t) - \sigma(\rho^2_t))^2 \leq 5\bar{\rho}_t$, last display is bounded from above by $\frac{5A}{2}\mathfrak{c}\|\bar{\rho}_t\|_2^2$. Putting all this together we conclude that

$$V'(t) \leq \sum_{k\geq 0} \Big(-\frac{(k\pi)^2}{a_k} + \frac{\mathfrak{c}}{2Aa_k} + \frac{5A}{2}\mathfrak{c} \Big) < \bar{\rho}_t, \psi_k >^2 .$$

Taking $A = 2(5\mathfrak{e})^{-1}$ we get $V'(t) \leq 2(\frac{5\mathfrak{e}^2}{4}+1)V(t)$ and we are done.
In the Neumann case the proof above can be adapted by observing that the set $\{\psi_k\}_k$ of eigenfunctions of Laplacian with Neumann boundary conditions is $\psi_k(u) = \sqrt{2}\cos(k\pi u)$. Observe that the previous proof also includes the case when $\mathfrak{e} = 0$, that is for the heat equation with Dirichlet or Neumann boundary conditions.

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